STABILITY OF THE STOCHASTIC HEAT EQUATION IN $L^1([0,1])$

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ABSTRACT. We consider the white-noise driven stochastic heat equation on $[0, \infty) \times [0, 1]$ with Lipschitz-continuous drift and diffusion coefficients b and σ. We derive an inequality for the $L^1([0,1])$-norm of the difference between two solutions. Using some martingale arguments, we show that this inequality provides some *a priori* estimates on solutions. This allows us to prove the strong existence and (partial) uniqueness of weak solutions when the initial condition belongs only to $L^1([0,1])$, and the stability of the solution with respect to this initial condition. We also obtain, under some conditions, some results concerning the large time behavior of solutions: uniqueness of the possible invariant distribution and asymptotic confluence of solutions.

1. INTRODUCTION AND RESULTS

1.1. **The equation.** Consider the stochastic heat equation with Neumann boundary conditions:

$$(1) \qquad \begin{cases} \partial_t u(t, x) & = & \partial_{xx} u(t, x) + b(u(t, x)) + \sigma(u(t, x))\dot{W}(t, x), \quad t \geq 0, \ x \in [0, 1], \\ u(0, x) & = & u_0(x), \quad x \in [0, 1], \\ \partial_x u(t, 0) & = & \partial_x u(t, 1) = 0, \quad t > 0. \end{cases}$$

Here $b, \sigma : \mathbb{R} \mapsto \mathbb{R}$ are the drift and diffusion coefficients and $u_0 : [0, 1] \mapsto \mathbb{R}$ is the initial condition. We write formally $W(dt, dx) = \dot{W}(t, x)dtdx$, for $W(dt, dx)$ a white noise on $[0, \infty) \times [0, 1]$ based on $dtdx$, see Walsh [13]. We will always assume in this paper that b, σ are Lipschitz-continuous, that is for some C,

$$(\mathcal{H}) \qquad \qquad \text{for all } r, z \in \mathbb{R}, \quad |b(r) - b(z)| + |\sigma(r) - \sigma(z)| \leq C|r - z|.$$

Our goals in this paper are the following:
• prove a strong existence and (partial) uniqueness result when the initial condition u_0 only belongs to $L^1([0,1])$ and some stability results of the solution with respect to such an initial condition;
• study the uniqueness of invariant measures and the asymptotic confluence of solutions.

We will investigate these two points by using some *a priori* estimates on the difference between two solutions u, v, obtained as a martingale dissipation of the $L^1([0,1])$-norm of $u(t) - v(t)$.

Let us mention that our results extend without difficulty to the case of Dirichlet boundary conditions and to the case of the unbounded domain $\mathbb{R}$ (with $u_0 \in L^1(\mathbb{R})$).

This equation has been much investigated, in particular since the work of Walsh [13]. In [13], one can find definitions of weak solutions, existence and uniqueness results, as well as proofs that solutions are Hölder-continuous, enjoy a Markov property, etc. Let us mention for example the works of Bally-Gyongy-Pardoux [1] (existence of solutions when the drift is only measurable), Gatarek-Goldys [7] (existence of solutions in law), Donati-Pardoux (comparison results and reflection problems), Bally-Pardoux (smoothness of the law of the solution), Bally-Millet-Sanz [3] (support theorem), etc. Sowers [12], Mueller [9] and Cerrai [4] have obtained some results on the invariant distributions and convergence to equilibrium.

1.2. **Weak solutions.** We will consider two types of *weak* solutions, which we now precisely define, following the ideas of Walsh [13]. When we refer to predictability, this is with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W, that is $\mathcal{F}_t = \sigma(W(A), A \in \mathcal{B}([0,t] \times [0,1]))$.

We denote by $L^p([0,1])$ the set of all measurable functions $f : [0,1] \mapsto \mathbb{R}$ such that $||f||_{L^p([0,1])} = (\int_0^1 |f(x)|^p dx)^{1/p} < \infty$.

Finally, we denote by $G_t(x,y)$ the Green kernel associated with the heat equation $\partial_t u = \partial_{xx} u$ on $\mathbb{R}_+ \times [0,1]$ with Neumann boundary conditions, whose explicit form can be found in Walsh [13]. Here we will only use that for some C_T, for all $x, y \in [0,1]$, all $t \in [0,T]$, see [13],

$$(2) \qquad\qquad 0 \leq G_t(x,y) \leq \frac{C_T}{\sqrt{t}} e^{-|x-y|^2/4t}.$$

Definition 1. *Assume* $(\mathcal{H})$*, and consider a* $\mathbb{R}$*-valued predictable process* $u = (u(t,x))_{t\geq 0, x\in[0,1]}$*.*
(i) For $u_0 \in L^1([0,1])$*,* u *is said to be a* **weak** *solution to (1) starting from* u_0 *if a.s.,*

$$(3) \qquad\qquad \text{for all } T > 0, \quad \sup_{[0,T]} ||u(t)||_{L^1([0,1])} + \int_0^T ||\sigma(u(t))||^2_{L^2([0,1])} dt < \infty$$

and if for all $\varphi \in C_b^2([0,1])$ *such that* $\varphi'(0) = \varphi'(1) = 0$*, for all* $t \geq 0$*, a.s.,*

$$(4) \qquad \int_0^1 u(t,x)\varphi(x)dx \;=\; \int_0^1 u_0(x)\varphi(x)dx + \int_0^t \int_0^1 \sigma(u(s,x))\varphi(x)W(ds,dx)$$
$$+ \int_0^t \int_0^1 [u(s,x)\varphi''(x) + b(u(s,x))\varphi(x)]dxds.$$

(ii) For u_0 *bounded-measurable,* u *is said to be a* **mild** *solution to (1) starting from* u_0 *if a.s.,*

$$(5) \qquad\qquad \text{for all } T > 0, \quad \sup_{[0,T]\times[0,1]} |u(t,x)| < \infty$$

and if for all $t \geq 0$*, all* $x \in [0,1]$*, a.s.,*

$$(6) \qquad u(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)[\sigma(u(s,y))W(ds,dy) + b(u(s,y))dyds].$$

Let us make a few comments. Recall that for $(H(s,y))_{s\geq 0, y\in[0,1]}$ a $\mathbb{R}$-valued predictable process, the stochastic integral $\int_0^t \int_0^1 H(s,y)W(ds,dy)$ is well-defined if and only if $\int_0^t \int_0^1 H^2(s,y)dyds < \infty$ a.s.

• Thus (3) implies that all the terms in (4) are well-defined. Clearly, condition (3) is not far from minimal.

• Next, (5) and (2) imply that all the terms in (6) are well-defined, but here (5) is clearly far from optimal.

When u_0 only belongs to $L^1([0,1])$, we will only be able to prove that (3) holds.

Let us finally recall that Walsh [13] proved, under $(\mathcal{H})$, that for any bounded-measurable initial condition u_0, there exists a unique mild solution u to (1), which is also a weak solution and which furthermore satisfies, for all $p \geq 1$, all $T > 0$, $\mathbb{E}[\sup_{[0,T]\times[0,1]} |u(t,x)|^p] < \infty$.

1.3. **Existence and stability in** $L^1([0,1])$**.** Our first goal is to extend the existence theory to more general initial conditions.

Theorem 2. *Assume* $(\mathcal{H})$*.*
(i) For $u_0 \in L^1([0,1])$*, there exists a weak solution* u *to (1) starting from* u_0*.*
(ii) This solution is unique in the following sense: for any sequence of bounded-measurable functions $u_0^n : [0,1] \mapsto \mathbb{R}$ *such that* $\lim_n ||u_0^n - u_0||_{L^1([0,1])} = 0$*, the sequence* $\sup_{[0,T]} ||u^n(t) - u(t)||_{L^1([0,1])}$ *tends to* 0 *in probability for any* T*. Here* u^n *is the unique mild solution to (1) starting from* u_0^n*.*

(iii) For $u_0, v_0 \in L^1([0,1])$, consider the two weak solutions u and v to (1) starting from u_0 and v_0 built in (i). For all $\gamma \in (0,1)$, all $T \geq 0$, we have

$$\mathbb{E}\left[\sup_{[0,T]} ||u(t) - v(t)||_{L^1([0,1])}^\gamma + \left(\int_0^T ||\sigma(u(t)) - \sigma(v(t))||_{L^2([0,1])}^2 dt\right)^{\gamma/2}\right] \leq C_{b,\gamma,T} ||u_0 - v_0||_{L^1([0,1])}^\gamma,$$

where $C_{b,\gamma,T}$ depends only on b, γ, T.
(iv) Assume now that b is non-increasing. For $u_0, v_0 \in L^1([0,1])$, let u, v be the two weak solutions to (1) starting from u_0 and v_0 built in (i). For all $\gamma \in (0,1)$, we have

$$\mathbb{E}\bigg[\sup_{[0,\infty)} ||u(t) - v(t)||_{L^1([0,1])}^\gamma + \left(\int_0^\infty ||b(u(t)) - b(v(t))||_{L^1([0,1])} dt\right)^\gamma$$

$$+ \left(\int_0^\infty ||\sigma(u(t)) - \sigma(v(t))||_{L^2([0,1])}^2 dt\right)^{\gamma/2}\bigg] \leq C_\gamma ||u_0 - v_0||_{L^1([0,1])}^\gamma,$$

where C_γ depends only on γ.

Observe that this result contains a regularization property. For example if $\sigma(z) = z$, even if u_0 does not belong to $L^2([0,1])$, the weak solution satisfies (3) and in particular $\sigma(u(t)) = u(t) \in L^2([0,1])$ for a.e. $t > 0$. For the same reasons, the stability result (iii) provides a better estimate for a.e. $t > 0$ than for $t = 0$.

To our knowledge, Theorem 2 is the first result concerning $L^1([0,1])$ initial conditions. Many works concern bounded-measurable (or continuous) initial conditions, see Walsh [13], Bally-Gyongy-Pardoux [1], Cerrai [4]. Another abundant literature deals with the Hilbert case (initial conditions in $L^2([0,1])$), see Pardoux [10], Da Prato-Zabczyk [5], Gatarek-Goldys [7].

The present well-posedness result is quite satisfying, since the requirement that $u_0 \in L^1([0,1])$ is very weak and seems necessary for (4) to make sense.

1.4. **Large time behavior.** We now wish to study the uniqueness of invariant measures.

Definition 3. *A probability measure Q on $L^1([0,1])$ is said to be an invariant distribution for (1) if, for u_0 a $L^1([0,1])$-valued random variable with law Q independent of W, for u the weak solution to (1) starting from u_0 built in Theorem 2, $\mathcal{L}(u(t)) = Q$ for all $t \geq 0$.*

We have the following result.

Theorem 4. *Assume $(\mathcal{H})$, that b is non-increasing and that $(\sigma, b) : \mathbb{R} \mapsto \mathbb{R}^2$ is injective. Then (1) admits at most one invariant distribution.*

To prove the asymptotic confluence of solutions, we need to strengthen the injectivity assumption.

$(\mathcal{I})$ $\quad \begin{cases} \text{There is a strictly increasing convex function } \rho : \mathbb{R}_+ \mapsto \mathbb{R}_+ \text{ with } \rho(0) = 0 \text{ such that} \\ \text{for all } r, z \in \mathbb{R}, \quad |b(r) - b(z)| + |\sigma(r) - \sigma(z)|^2 \geq \rho(|r - z|). \end{cases}$

Theorem 5. *Assume $(\mathcal{H})$, that b is non-increasing and $(\mathcal{I})$.*
(i) The following asymptotic confluence property holds: for $u_0, v_0 \in L^1([0,1])$, for u, v the weak solutions to (1) starting from u_0 and v_0 built in Theorem 2,

$$a.s., \quad \lim_{t \to \infty} ||u(t) - v(t)||_{L^1([0,1])} = 0.$$

(ii) Assume additionally that (1) admits an invariant distribution Q. Then for $u_0 \in L^1([0,1])$, for u the corresponding weak solution to (1), $u(t)$ goes in law to Q as $t \to \infty$.

Clearly, $(\mathcal{I})$ holds if b is C^1 with $b' \leq -\epsilon < 0$ (choose $\rho(z) = \epsilon z$) or if σ is C^1 with $|\sigma'| \geq \epsilon > 0$ (choose $\rho(z) = (\epsilon z)^2$). One may also combine conditions on b and σ.

But $(\mathcal{I})$ also holds if b is C^1 and if $b' \leq 0$ vanishes reasonably. For example if $b(z) = -\mathrm{sg}(z) \min(|z|, |z|^p)$ for some $p \geq 1$, choose $\rho = \epsilon \rho_p$ with ϵ small enough and $\rho_p(z) = z^p$ for $z \in [0, 1]$ and $\rho_p(z) = pz - p + 1$ for $z \geq 1$. If $b(z) = -z - \sin z$, choose $\rho = \epsilon \rho_3$ with ϵ small enough.

One may also consider the case where σ is monotonous with σ' vanishing reasonably.

Let us now compare Theorems 4 and 5 with known results. The works cited below sometimes concern different boundary conditions, but we believe this is not important.

• Sowers [12] has proved the existence of an invariant distribution supported by $C([0, 1])$, assuming $(\mathcal{H})$, that σ is bounded and that b is of the form $b(z) = -\alpha z + f(z)$, for some bounded f and some $\alpha > 0$. He obtained uniqueness of this invariant distribution when σ is sufficiently small and bounded from below.

• Mueller [9] has obtained some surprising coupling results, implying in particular the uniqueness of an invariant distribution as well as a the trend to equilibrium. He assumes $(\mathcal{H})$, that σ is bounded from above and from below and that b is non-increasing, with $|b(z) - b(r)| \geq \alpha |z - r|$ for some $\alpha > 0$.

• Cerrai [4] assumed that σ is strictly monotonous (it may vanish, but only at one point).
(i) She obtained an asymptotic confluence result which we do not recall here and concerns, roughly, the case $b(z) \simeq -\mathrm{sg}(z)|z|^m$ as $z \to \pm\infty$, for some $m > 1$.
(ii) Assuming $(\mathcal{H})$, she proved uniqueness of the invariant distribution as well as an asymptotic confluence property, under the conditions that for all $r \leq z$, $b(z) - b(r) \leq \lambda(z - r)$, and $|\sigma(z) - \sigma(r)| \geq \mu|z - r|$, for some $\mu > 0$ and some $\lambda < \mu^2/2$ (if b is non-increasing, choose $\lambda = 0$).

Thus the main advantages of the present paper are that the uniqueness of the invariant measure requires very few conditions, and we allow σ to vanish (it may be compactly supported).

Example 1. Assume $(\mathcal{H})$ and that b strictly decreasing. Then there exists at most one invariant distribution. If $b(z) = -z$ or $b(z) = -z - \sin z$ or $b(z) = -\mathrm{sg}(z) \min(|z|, |z|^p)$ for some $p > 1$, then we have asymptotic confluence of solutions. Here to apply [12, 9] one needs to assume additionally that σ is bounded from above and from below, while to apply [4], one has to suppose that σ is strictly monotonous.

Example 2. Assume $(\mathcal{H})$, that b is non-increasing and that σ is strictly monotonous. Then there exists at most one invariant distribution.
If furthermore σ is C^1 with $0 < c < \sigma' < C$, then we get asymptotic confluence of solutions using [4] or Theorem 5 (here [12, 9] cannot apply, since σ vanishes). But now if $\sigma' \geq 0$ reasonably vanishes then Theorem 5 applies, which is not the case of [4]: take e.g. $\sigma(z) = \mathrm{sg}(z) \min(|z|, |z|^p)$ for some $p > 1$, or $\sigma(z) = z + \sin z$.

Example 3. Consider the compactly supported coefficient $\sigma(z) = (1 - z^2)\mathbf{1}_{\{|z| \leq 1\}}$. Assume that b is C^1, non-increasing, with $b'(z) \leq -\epsilon < 0$ for $z \in (-\infty, -1) \cup \{0\} \cup (1, +\infty)$. Then Theorems 4 and 5 apply, while [12, 9, 4] do not.
Observe here that if $b(z_0) = 0$ for some $z_0 \notin (-1, 1)$, then $u(t) \equiv z_0$ is the (unique) stationary solution. If now $b(-1) > 0$ and $b(1) < 0$, then the invariant measure Q (that exists due to Sowers [12]) is unique and one may show, using the comparison Theorem of Donati-Pardoux [6], that Q is supported by $[-1, 1]$-valued continuous functions on $[0, 1]$.

However, there are some cases where [12, 4] provide some better results than ours.

Example 4. If $\sigma(z) = \mu z$ and $b(z) = \lambda z$, then $u(t) \equiv 0$ is an obvious stationary solution. Theorems 4 and 5 apply if $\lambda \leq 0$ and $|\lambda| + |\mu| > 0$. Cerrai [4] was able to treat the case $\lambda > 0$ provided $\mu^2/2 > \lambda$.

Example 5. If σ is small enough and bounded from below and if $b(z) = -\alpha z + h(z)$, with $\alpha > 0$ and h bounded, then Sowers [12] obtains the uniqueness of the invariant distribution even if b is not non-increasing.

1.5. **Plan of the paper.** In the next section, we prove some inequalities concerning the $L^1([0,1])$-norm of the difference between any pair of *mild* solutions to (1). Section 3 is dedicated to the proof of our existence result Theorem 2. Theorems 4 and 5 are checked in Section 4. We briefly discuss the multi-dimensional equation in Section 5 and conclude the paper with an appendix containing technical results.

2. ON THE $L^1([0,1])$-NORM OF THE DIFFERENCE BETWEEN TWO MILD SOLUTIONS

All our study is based on the following result. We set $\mathrm{sg}(z) = 1$ for $z \geq 0$ and $\mathrm{sg}(z) = -1$ for $z < 0$.

Proposition 6. *Assume $(\mathcal{H})$. For two bounded-measurable initial conditions u_0, v_0, let u, v be the corresponding mild solutions to (1). Then, enlarging the probability space if necessary, there is a Brownian motion $(B_t)_{t\geq 0}$ such that a.s., for all $t \geq 0$,*

$$
(7) \qquad \|u(t) - v(t)\|_{L^1([0,1])} \leq \|u_0 - v_0\|_{L^1([0,1])} + \int_0^t \|\sigma(u(s)) - \sigma(v(s))\|_{L^2([0,1])} dB_s
$$
$$
+ \int_0^t \int_0^1 \mathrm{sg}(u(s,x) - v(s,x))(b(u(s,x)) - b(v(s,x))) dx ds.
$$

Proof. We divide the proof into several steps, following closely the ideas of Donati-Pardoux [6, Theorem 2.1], to which we refer for technical details.

Step 1. Consider an orthonormal basis $(e_k)_{k\geq 1}$ of $L^2([0,1])$. For $k \geq 1$, we set $B_t^k = \int_0^t \int_0^1 e_k(x) W(ds, dx)$. Then $(B^k)_{k\geq 1}$ is a family of independent Brownian motions. For $n \geq 1$, consider the unique adapted solution $u^n \in L^2(\Omega \times [0,T], V)$, where $V = \{f \in H^1([0,1]), f'(0) = f'(1) = 0\}$, to

$$
u^n(t,x) = u_0(x) + \int_0^t \left[\partial_{xx} u^n(s,x) ds + b(u^n(s,x))\right] ds + \sum_{k=1}^n \int_0^t \sigma(u^n(s,x)) e_k(x) dB_s^k.
$$

We refer to Pardoux [10] for existence, uniqueness and properties of this solution. We also consider the solution v^n to the same equation starting from v_0. Then, as shown in [6],

$$
(8) \qquad \lim_n \sup_{[0,T]\times[0,1]} \mathbb{E}[|u^n(t,x) - u(t,x)|^2 + |v^n(t,x) - v(t,x)|^2] = 0.
$$

Step 2. For $\epsilon > 0$, we introduce a nonnegative C^2 function ϕ_ϵ such that $\phi_\epsilon(z) = |z|$ for $|z| \geq \epsilon$, with $|\phi'_\epsilon(z)| \leq 1$ and $0 \leq \phi''_\epsilon(z) \leq 2\epsilon^{-1} \mathbf{1}_{|z| < \epsilon}$. When applying the Itô formula (see [6] for details), we get

$$
(9) \qquad \int_0^1 \phi_\epsilon(u^n(t,x) - v^n(t,x)) dx = \int_0^1 \phi_\epsilon(u_0(x) - v_0(x)) dx
$$
$$
+ \int_0^t \int_0^1 \phi'_\epsilon(u^n(s,x) - v^n(s,x)) \partial_{xx}[u^n(s,x) - v^n(s,x)] dx ds
$$
$$
+ \int_0^t \int_0^1 \phi'_\epsilon(u^n(s,x) - v^n(s,x))[b(u^n(s,x)) - b(v^n(s,x))] dx ds
$$
$$
+ \sum_{k=1}^n \int_0^t \int_0^1 \phi'_\epsilon(u^n(s,x) - v^n(s,x))[\sigma(u^n(s,x)) - \sigma(v^n(s,x))] e_k(x) dx dB_s^k
$$
$$
+ \frac{1}{2} \sum_{k=1}^n \int_0^t \int_0^1 \phi''_\epsilon(u^n(s,x) - v^n(s,x))[\sigma(u^n(s,x)) - \sigma(v^n(s,x))]^2 e_k^2(x) dx ds
$$
$$
=: I_\epsilon^1 + I_\epsilon^2(t) + I_\epsilon^3(t) + I_\epsilon^4(t) + I_\epsilon^5(t).
$$

Since $|z| \leq \phi_\epsilon(z) \leq |z| + \epsilon$ for all z, we easily get, a.s.,

$$
\lim_{\epsilon \to 0} \int_0^1 \phi_\epsilon(u^n(t,x) - v^n(t,x)) dx = \|u^n(t) - v^n(t)\|_{L^1([0,1])} \quad \text{and} \quad \lim_{\epsilon \to 0} I_\epsilon^1 = \|u_0 - v_0\|_{L^1([0,1])}.
$$

An integration by parts, using that $\partial_x[u^n(t,0) - v^n(t,0)] = \partial_x[u^n(t,1) - v^n(t,1)] = 0$ shows that

$$I_\epsilon^2(t) = -\int_0^t \int_0^1 \phi_\epsilon''(u^n(s,x) - v^n(s,x))[\partial_x(u^n(s,x) - v^n(s,x))]^2 \leq 0.$$

Since $\phi_\epsilon''(z-r)(\sigma(z) - \sigma(r))^2 \leq C\epsilon^{-1}\mathbf{1}_{|z-r|\leq\epsilon}|z-r|^2 \leq C\epsilon$ by $(\mathcal{H})$, we have $I_\epsilon^5(t) \leq Cnt\epsilon$, whence

$$\lim_{\epsilon\to 0} I_\epsilon^5(t) = 0 \text{ a.s.}$$

Using that $|\phi_\epsilon'(z) - \mathrm{sg}(z)| \leq \mathbf{1}_{\{|z|\leq\epsilon\}}$ and $(\mathcal{H})$, one obtains a.s.

$$\lim_{\epsilon\to 0}\left|I_\epsilon^3(t) - \int_0^t\int_0^1 \mathrm{sg}(u^n(s,x) - v^n(s,x))(b(u^n(s,x)) - b(v^n(s,x)))dxds\right|$$
$$\leq \lim_{\epsilon\to 0}\int_0^t\int_0^1 \mathbf{1}_{|u^n(s,x)-v^n(s,x)|\leq\epsilon}|b(u^n(s,x)) - b(v^n(s,x))|dxds \leq \lim_{\epsilon\to 0} Ct\epsilon = 0.$$

Similarly,

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\left(I_\epsilon^4(t) - \sum_{k=1}^n\int_0^t\int_0^1 \mathrm{sg}(u^n(s,x) - v^n(s,x))[\sigma(u^n(s,x)) - \sigma(v^n(s,x))]e_k(x)dxdB_s^k\right)^2\right] = 0.$$

Thus we can pass to the limit as $\epsilon \to 0$ in (9) and get, a.s.,

$$\begin{aligned}||u^n(t) - v^n(t)||_{L^1([0,1])} &\leq ||u_0 - v_0||_{L^1([0,1])} \\ &+ \int_0^t\int_0^1 \mathrm{sg}(u^n(s,x) - v^n(s,x))[b(u^n(s,x)) - b(v^n(s,x))]dxds \\ (10)\qquad &+ \sum_{k=1}^n\int_0^t\int_0^1 \mathrm{sg}(u^n(s,x) - v^n(s,x))[\sigma(u^n(s,x)) - \sigma(v^n(s,x))]e_k(x)dxdB_s^k.\end{aligned}$$

Step 3. Using $(\mathcal{H})$, there holds, for all r_1, z_1, r_2, z_2 in $\mathbb{R}$,

$$(11)\qquad \left|\mathrm{sg}(r_1 - z_1)[\sigma(r_1) - \sigma(z_1)] - \mathrm{sg}(r_2 - z_2)[\sigma(r_2) - \sigma(z_2)]\right| \leq C(|r_1 - r_2| + |z_1 - z_2|),$$
$$(12)\qquad \left|\mathrm{sg}(r_1 - z_1)[b(r_1) - b(z_1)] - \mathrm{sg}(r_2 - z_2)[b(r_2) - b(z_2)]\right| \leq C(|r_1 - r_2| + |z_1 - z_2|).$$

Indeed, it suffices, by symmetry, to check that $\left|\mathrm{sg}(r_1 - z_1)[\sigma(r_1) - \sigma(z_1)] - \mathrm{sg}(r_2 - z_1)[\sigma(r_2) - \sigma(z_1)]\right| \leq C|r_1 - r_2|$. If $\mathrm{sg}(r_1 - z_1) = \mathrm{sg}(r_2 - z_2)$, this is obvious. If now $r_1 \leq z_1 \leq r_2$ (or $r_1 \geq z_1 \geq r_2$) we get the upper-bound $|\sigma(r_1) + \sigma(r_2) - 2\sigma(z_1)| \leq C(|r_1 - z_1| + |r_2 - z_1|) = C|r_1 - r_2|$.
Using (8), it is thus routine to make n tend to infinity in (10) and to obtain, a.s.,

$$\begin{aligned}||u(t) - v(t)||_{L^1([0,1])} &\leq ||u_0 - v_0||_{L^1([0,1])} + \int_0^t\int_0^1 \mathrm{sg}(u(s,x) - v(s,x))[b(u(s,x)) - b(v(s,x))]dxds \\ (13)\qquad &+ \sum_{k=1}^\infty\int_0^t\int_0^1 \mathrm{sg}(u(s,x) - v(s,x))[\sigma(u(s,x)) - \sigma(v(s,x))]e_k(x)dxdB_s^k.\end{aligned}$$

For the last term, we used that, by the Plancherel identity, setting for simplicity

$$\begin{aligned}\alpha_n(s,x) &= \mathrm{sg}(u^n(s,x) - v^n(s,x))[\sigma(u^n(s,x)) - \sigma(v^n(s,x))], \\ \alpha(s,x) &= \mathrm{sg}(u(s,x) - v(s,x))[\sigma(u(s,x)) - \sigma(v(s,x))],\end{aligned}$$

there holds

$$
\mathbb{E}\Big[\Big(\sum_{k=1}^{n}\int_0^t\int_0^1\alpha_n(s,x)e_k(x)dB_s^k-\sum_{k=1}^{\infty}\int_0^t\int_0^1\alpha(s,x)e_k(x)dB_s^k\Big)^2\Big]
$$

$$
\leq \int_0^t\mathbb{E}\Big[\sum_{k\geq 1}\Big(\int_0^1\big\{\alpha_n(s,x)-\alpha(s,x)\big\}e_k(x)dx\Big)^2\Big]ds+\sum_{k\geq n+1}\int_0^t\mathbb{E}\Big[\Big(\int_0^1\alpha(s,x)e_k(x)dx\Big)^2\Big]ds
$$

$$
\leq \int_0^t\mathbb{E}\Big[||\alpha_n(s)-\alpha(s)||^2_{L^2([0,1])}\Big]ds+\sum_{k\geq n+1}\int_0^t\mathbb{E}\Big[\Big(\int_0^1\alpha(s,x)e_k(x)dx\Big)^2\Big]ds =: I_n(t)+J_n(t).
$$

Using (11) and then (8), $I_n(t)\leq C\int_0^t\int_0^1\mathbb{E}[|u^n(s,x)-u(s,x)|^2+|v^n(s,x)-v(s,x)|^2]dxds$ tends to 0 as $n\to\infty$. Finally, $J_n(t)$ tends to 0 because $\sum_{k\geq 1}\int_0^t\mathbb{E}[(\int_0^1\alpha(s,x)e_k(x)dx)^2]ds=\int_0^t\mathbb{E}[||\alpha(s)||^2_{L^2([0,1])}]ds\leq C\int_0^t\int_0^1\mathbb{E}(|u(s,x)-v(s,x)|^2)dxds<\infty$.

Step 4. A standard representation argument (see e.g. Revuz-Yor [11, Proposition 3.8 and Theorem 3.9 p 202-203]) concludes the proof, because the last term on the RHS of (13) is a continuous local martingale with bracket

$$
\int_0^t\sum_{k=1}^{\infty}\Big(\int_0^1 \mathrm{sg}(u(s,x)-v(s,x))[\sigma(u(s,x))-\sigma(v(s,x))]e_k(x)dx\Big)^2 ds = \int_0^t||\sigma(u(s))-\sigma(v(s))||^2_{L^2([0,1])}ds.
$$

We used here again that $(e_k)_{k\geq 1}$ is an orthonormal basis of $L^2([0,1])$. $\qquad\square$

Corollary 7. *Adopt the notation and assumptions of Proposition 6. For all $\gamma\in(0,1)$, all $T\geq 0$,*

$$
\mathbb{E}\left[\sup_{[0,T]}||u(t)-v(t)||^{\gamma}_{L^1([0,1])}+\Big(\int_0^T||\sigma(u(t))-\sigma(v(t))||^2_{L^2([0,1])}dt\Big)^{\gamma/2}\right]\leq C_{b,\gamma,T}||u_0-v_0||^{\gamma}_{L^1([0,1])},
$$

where $C_{b,\gamma,T}$ depends only on b,γ,T.

Proof. Let C be the Lipschitz constant of b. Denote by L_t the RHS of (7). The Itô formula yields

$$
||u(t)-v(t)||_{L^1([0,1])}e^{-Ct} \leq L_t e^{-Ct}
$$

$$
= ||u_0-v_0||_{L^1([0,1])}-C\int_0^t e^{-Cs}L_s ds
$$

$$
+\int_0^t||\sigma(u(s))-\sigma(v(s))||_{L^2([0,1])}e^{-Cs}dB_s
$$

$$
+\int_0^t\int_0^1 e^{-Cs}\mathrm{sg}(u(s,x)-v(s,x))(b(u(s,x))-b(v(s,x)))dxds.
$$

But $\int_0^1\mathrm{sg}(u(s,x)-v(s,x))(b(u(s,x))-b(v(s,x)))dx\leq C||u(s)-v(s)||_{L^1([0,1])}\leq CL_s$. Hence

$$
||u(t)-v(t)||_{L^1([0,1])}e^{-Ct} \leq ||u_0-v_0||_{L^1([0,1])}+\int_0^t||\sigma(u(s))-\sigma(v(s))||_{L^2([0,1])}e^{-Cs}dB_s =: M_t.
$$

Hence M_t is a nonnegative local martingale with bracket $\langle M\rangle_t=\int_0^t||\sigma(u(s))-\sigma(v(s))||^2_{L^2([0,1])}e^{-2Cs}ds$. Applying Lemma 9, we immediately get, for $\gamma\in(0,1)$,

$$
\mathbb{E}\left[\sup_{[0,\infty)}||u(t)-v(t)||^{\gamma}_{L^1([0,1])}e^{-C\gamma t}+\Big(\int_0^{\infty}||\sigma(u(s))-\sigma(v(s))||^2_{L^2([0,1])}e^{-2Cs}ds\Big)^{\gamma/2}\right]
$$

$$
\leq C_{\gamma}||u_0-v_0||^{\gamma}_{L^1([0,1])}.
$$

The result easily follows. $\qquad\square$

Finally, one can say a little more when b is non-increasing.

Corollary 8. *Adopt the notation and assumptions of Proposition 6 and assume that b is non-increasing. Then for all $\gamma \in (0,1)$,*

$$\mathbb{E}\Big[\sup_{[0,\infty)} ||u(t) - v(t)||^{\gamma}_{L^1([0,1])} + \Big(\int_0^{\infty} ||b(u(t)) - b(v(t))||_{L^1([0,1])} dt \Big)^{\gamma}$$

$$+ \Big(\int_0^{\infty} ||\sigma(u(t)) - \sigma(v(t))||^2_{L^2([0,1])} dt \Big)^{\gamma/2} \Big] \quad \leq \quad C_{\gamma} ||u_0 - v_0||^{\gamma}_{L^1([0,1])},$$

where C_{γ} depends only on γ.

Proof. Since b is non-increasing, Proposition 6 yields

$$||u(t) - v(t)||_{L^1([0,1])} + \int_0^t ||b(u(s)) - b(v(s))||_{L^1([0,1])} ds$$

$$\leq \quad ||u_0 - v_0||_{L^1([0,1])} + \int_0^t ||\sigma(u(s)) - \sigma(v(s))||_{L^2([0,1])} dB_s =: M_t,$$

which is thus a nonnegative martingale with bracket $\langle M \rangle_t = \int_0^t ||\sigma(u(s)) - \sigma(v(s))||^2_{L^2([0,1])} ds$. Lemma 9 allows us to conclude. $\qquad\square$

3. Existence theory in $L^1([0,1])$

The goal of this section is to give the

Proof of Theorem 2. We start with point (i). Let thus $u_0 \in L^1([0,1])$ and consider a sequence of bounded-measurable initial conditions $(u_0^n)_{n\geq 1}$ such that $||u_0^n - u_0||_{L^1([0,1])} \leq 2^{-n}$. For each $n \geq 1$, denote by u^n the mild solution to (1) starting from u_0^n. Using Corollary 7 (with $\gamma = 1/2$), we deduce that a.s.,

$$\sum_{n\geq 1} \left[\sup_{[0,T]} ||u^{n+1}(t) - u^n(t)||^{1/2}_{L^1([0,1])} + \Big(\int_0^T ||\sigma(u^{n+1}(t)) - \sigma(u^n(t))||^2_{L^2([0,1])} dt \Big)^{1/4} \right] < \infty,$$

which implies that

$$\sum_{n\geq 1} \left[\sup_{[0,T]} ||u^{n+1}(t) - u^n(t)||_{L^1([0,1])} + ||\sigma(u^{n+1}) - \sigma(u^n)||_{L^2([0,T]\times[0,1])} \right] < \infty.$$

Using some completeness arguments, we deduce that there are some (predictable) processes u and S such that a.s., for all $T > 0$, $\sup_{[0,T]} ||u(t)||_{L^1([0,1])} + \int_0^T ||S(t)||^2_{L^2([0,1])} dt < \infty$ and

$$\lim_n \sup_{[0,T]} ||u(t) - u^n(t)||_{L^1([0,1])} = 0, \quad \lim_n ||S - \sigma(u^n)||_{L^2([0,T]\times[0,1])} = 0.$$

Since σ is Lipschitz-continuous, we deduce from the first equality that $\lim_n ||\sigma(u) - \sigma(u^n)||_{L^1([0,T]\times[0,1])} = 0$, while from the second one, $\lim_n ||S - \sigma(u^n)||_{L^1([0,T]\times[0,1])} = 0$. Consequently, $S = \sigma(u)$ a.e. and we finally conclude that a.s.,

$$(14) \qquad \text{for all } T > 0, \quad \lim_n \Big(\sup_{[0,T]} ||u(t) - u^n(t)||_{L^1([0,1])} + \int_0^T ||\sigma(u(t)) - \sigma(u^n(t))||^2_{L^2([0,1])} dt \Big) = 0.$$

It remains to prove that u is a weak solution to (1). We have already seen that u satisfies (3). Next, for $\varphi \in C_b^2([0,1])$ with $\varphi'(0) = \varphi'(1) = 0$, for $t \geq 0$, we know that a.s., $A_t^{n,\varphi} = B_t^{n,\varphi}$ for all $n \geq 1$, where

$$A_t^{n,\varphi} \quad := \quad \int_0^1 \varphi(x) u^n(t,x) dx - \int_0^1 \varphi(x) u_0^n(x) dx - \int_0^t \int_0^1 [u^n(s,x)\varphi''(x) + b(u^n(s,x))\varphi(x)] dx ds$$

$$B_t^{n,\varphi} \quad := \quad \int_0^t \int_0^1 \sigma(u^n(s,x))\varphi(x) W(ds,dx).$$

It directly follows from (14) and ($\mathcal{H}$) that a.s.,

$$\lim_{n\to\infty} A_t^{n,\varphi} = \int_0^1 \varphi(x)u(t,x)dx - \int_0^1 \varphi(x)u_0(x)dx - \int_0^t \int_0^1 [u(s,x)\varphi''(x) + b(u(s,x))\varphi(x)]dxds.$$

We deduce that $B_t^\varphi := \lim_n B_t^{n,\varphi}$ exists a.s., and it only remains to check that $B_t^\varphi = C_t^\varphi$ a.s., where $C_t^\varphi := \int_0^t \int_0^1 \sigma(u(s,x))\varphi(x)W(ds,dx)$. To this end, consider, for $M > 0$, the stopping time

$$\tau_M = \inf\left\{ r \geq 0, \ \int_0^r ||\sigma(u(s))||_{L^2([0,1])}^2 ds + \sup_n \int_0^r ||\sigma(u^n(s))||_{L^2([0,1])}^2 ds \geq M \right\}.$$

Using (14) and the dominated convergence Theorem, we see that for each $M > 0$,

$$\lim_n \mathbb{E}[|B_{t\wedge\tau_M}^{n,\varphi} - C_{t\wedge\tau_M}^\varphi|^2] = \lim_n \mathbb{E}\left[\int_0^{t\wedge\tau_M} ||(\sigma(u(s)) - \sigma(u^n(s))\varphi||_{L^2([0,1])}^2 ds \right] = 0.$$

But we also deduce from (14) that a.s., $\sup_n \int_0^T ||\sigma(u^n(s))||_{L^2([0,1])}^2 ds < \infty$ for all $T > 0$, whence $\lim_{M\to\infty} \tau_M = \infty$ a.s. We easily conclude that $B_t^{n,\varphi}$ tends to C_t^φ in probability, whence $B_t^\varphi = C_t^\varphi$ a.s.

Point (ii) is easily checked: let $(\tilde{u}_0^n)_{n\geq 1}$ be another sequence of bounded-measurable initial conditions converging to u_0 and let $(\tilde{u}^n)_{n\geq 1}$ be the corresponding sequence of mild solutions to (1). Then necessarily, $||u_0^n - \tilde{u}_0^n||_{L^1([0,1])}$ tends to 0, whence, by Corollary 7, $\sup_{[0,T]} ||u^n(t) - \tilde{u}^n(t)||_{L^1([0,1])}$ tends also to 0, in probability. Using (14), we conclude that $\sup_{[0,T]} ||u(t) - \tilde{u}^n(t)||_{L^1([0,1])}$ tends to 0 in probability.

We now prove point (iii). For u_0 and v_0 in $L^1([0,1])$, we consider u_0^n and v_0^n bounded-measurable with $||u_0^n - u_0||_{L^1([0,1])} + ||v_0^n - v_0||_{L^1([0,1])} \leq 2^{-n}$. We denote by u,v,u^n,v^n the corresponding weak solutions to (1). In the proof of (i), we have seen that a.s., $\lim_n \sup_{[0,T]}[||u^n(t) - u(t)||_{L^1([0,1])} + ||v^n(t) - v(t)||_{L^1([0,1])}] = 0$ and $\lim_n \int_0^T [||\sigma(u^n(t)) - \sigma(u(t))||_{L^2([0,1])}^2 + ||\sigma(v^n(t)) - \sigma(v(t))||_{L^2([0,1])}^2]dt = 0$. Using the Fatou Lemma and Corollary 7, we thus get

$$\mathbb{E}\left[\sup_{[0,T]} ||u(t) - v(t)||_{L^1([0,1])}^\gamma + \left(\int_0^T ||\sigma(u(t)) - \sigma(v(t))||_{L^2([0,1])}^2 dt \right)^{\gamma/2} \right]$$

$$\leq \liminf_n \mathbb{E}\left[\sup_{[0,T]} ||u^n(t) - v^n(t)||_{L^1([0,1])}^\gamma + \left(\int_0^T ||\sigma(u^n(t)) - \sigma(v^n(t))||_{L^2([0,1])}^2 dt \right)^{\gamma/2} \right]$$

$$\leq \liminf_n C_{\gamma,T} ||u_0^n - v_0^n||_{L^1([0,1])}^\gamma = C_{\gamma,T} ||u_0 - v_0||_{L^1([0,1])}^\gamma.$$

Point (iv) is checked similarly. $\qquad\square$

4. Large time behavior

We now prove the uniqueness of the invariant measure.

Proof of Theorem 4. Consider two invariant distributions Q and $\tilde{Q}$ for (1), see Definition 3. Let u_0 be Q-distributed and $\tilde{u}_0$ be $\tilde{Q}$-distributed. Consider the corresponding (stationary) weak solutions $u,\tilde{u}$ to (1). Applying Theorem 2-(iv) and the Cauchy-Schwarz inequality, $\int_0^\infty K_s ds < \infty$ a.s., where

$$K_s := K(u(s),\tilde{u}(s)) = ||b(u(s)) - b(\tilde{u}(s))||_{L^1([0,1])} + ||\sigma(u(s)) - \sigma(\tilde{u}(s))||_{L^1([0,1])}^2.$$

Using Lemma 10, there is a sequence $(t_n)_{n\geq 1}$ such that K_{t_n} tends to 0 in probability. Consider the function $\phi(r) = r/(1+r)$ on $\mathbb{R}_+$, and define $\Psi : L^1([0,1]) \times L^1([0,1]) \mapsto \mathbb{R}_+$ as $\Psi(f,g) = \phi(K(f,g))$. Then $\lim_n \mathbb{E}[\Psi(u(t_n),v(t_n))] = \lim_n \mathbb{E}[\phi(K_{t_n})] = 0$.

We now apply Lemma 11. The space $L^1([0,1])$ is Polish and for each $n \geq 1$, $\mathcal{L}(u(t_n)) = Q$ and $\mathcal{L}(\tilde{u}(t_n)) = \tilde{Q}$. The function Ψ is clearly continuous on $L^1([0,1]) \times L^1([0,1])$, (because σ, b are Lipschitz-continuous). Finally, $\Psi(f,g) > 0$ for all $f \neq g$ (because $\Psi(f,g) = 0$ implies that $b \circ f = b \circ g$ and $\sigma \circ f = \sigma \circ g$ a.e., whence $f = g$ a.e. since (σ,b) is injective). Lemma 11 thus yields $Q = \tilde{Q}$. $\qquad\square$

Finally, we give the

Proof of Theorem 5. Point (ii) is immediately deduced from point (i). Let thus $u_0, v_0 \in L^1([0,1])$ be fixed and let u, v be the corresponding weak solutions to (1). We know from $(\mathcal{I})$, the Jensen inequality and Theorem 2-(iv) that a.s.,

$$\int_0^\infty \rho(||u(t) - v(t)||_{L^1([0,1])})dt \leq \int_0^\infty ||\rho(|u(t) - v(t)|)||_{L^1([0,1])}dt$$

$$\leq \int_0^\infty \left|\left||b(u(t)) - b(v(t))| + |\sigma(u(t)) - \sigma(v(t))|^2\right|\right|_{L^1([0,1])}dt < \infty.$$

Using Lemma 10, one may thus find an increasing sequence $(t_n)_{n \geq 1}$ such that $\rho(||u(t_n) - v(t_n)||_{L^1([0,1])})$ tends to 0 in probability, so that $||u(t_n) - v(t_n)||_{L^1([0,1])}$ also tends to 0 in probability (because due to $\mathcal{I}$, ρ is strictly increasing and vanishes only at 0). Next, we use Theorem 2-(iv) with e.g. $\gamma = 1/2$ to get, setting $\Delta_t = \sup_{[t,\infty)} ||u(s) - v(s)||_{L^1([0,1])}$,

$$\mathbb{E}\left[\Delta_{t_n}^{1/2}\Big| \mathcal{F}_{t_n}\right] \leq C||u(t_n) - v(t_n)||_{L^1([0,1])}^{1/2} \to 0 \text{ in probability.}$$

We used here that conditionally on $\mathcal{F}_{t_n}$, $(u(t_n + t, x))_{t \geq 0, x \in [0,1]}$ is a weak solution to (1), starting from $u(t_n)$ (with a translated white noise). Thus for any $\epsilon > 0$, using the Markov inequality

$$P[\Delta_{t_n} > \epsilon] = \mathbb{E}[P(\Delta_{t_n} > \epsilon | \mathcal{F}_{t_n})] \leq \mathbb{E}\left[\min\left(1, \epsilon^{-1/2}\mathbb{E}\left[\Delta_{t_n}^{1/2}\Big| \mathcal{F}_{t_n}\right]\right)\right],$$

which tends to 0 as $n \to \infty$ by dominated convergence. Consequently, as n tends to infinity,

(15) Δ_{t_n} tends to 0 in probability.

But a.s. $s \mapsto \Delta_s = \sup_{[s,\infty)} ||u(t) - v(t)||_{L^1([0,1])}$ is non-increasing, and thus admits a limit as $s \to \infty$, which can be only 0 due to (15).   □

5. Toward the multi-dimensional case?

Consider now a bounded smooth domain $D \subset \mathbb{R}^d$, for some $d \geq 2$. Consider the (scalar) equation

(16) $\partial_t u(t, x) = \Delta u(t, x) + b(u(t, x)) + \sigma(u(t, x))\dot{W}(t, x), \quad t \geq 0, \ x \in D,$

with some Neumann boundary condition. Here $W(dt, dx) = \dot{W}(t, x)dtdx$ is a white noise on $[0, \infty) \times D$ based on $dtdx$. We assume that $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz-continuous.

It is well known that the mild equation makes no sense in such a case, since even if $\sigma(u)$ is bounded, $G_{t-s}(x, y)\sigma(u(s, y))$ does not belong to $L^2([0, t] \times D)$. The existence of solutions is thus still an open problem. See however Walsh [13] when $\sigma \equiv 1$, $b(u) = \alpha u$ and Nualart-Rozovskii [8] when $\sigma(u) = u$, $b(u) = \alpha u$. In these works, the authors manage to define some *ad-hoc* notion of solutions, using that the equations can be solved more or less explicitly. In the literature, one almost always considers the simpler case where the noise W is colored, see Da Prato-Zabczyk [5].

However the weak form makes sense: a predictable process $u = (u(t, x))_{t \geq 0, x \in D}$ is a weak solution if a.s.,

(17)   for all $T > 0$, $\sup_{[0,T]} ||u(t)||_{L^1(D)} + \int_0^T ||\sigma(u(t))||_{L^2(D)}^2 dt < \infty$

and if for all function $\varphi \in C_b^2(D)$ (with Neumann conditions on ∂D), all $t \geq 0$, a.s.,

$$\int_D u(t, x)\varphi(x)dx = \int_D u_0(x)\varphi(x)dx + \int_0^t \int_D [\{u(s, x)\Delta\varphi(x) + b(u(s, x))\}dxds + \sigma(u(s, x))\varphi(x)W(ds, dx)].$$

Assume now that $\sigma(0) = b(0) = 0$. Then $v \equiv 0$ is a weak solution. Furthermore, the estimate of Theorem 2-(iii) *a priori* holds. Choosing $u_0 \in L^1(D)$ and $v_0 = 0$, this would imply (17). Unfortunately, we are not able to make this *a priori* estimate rigorous.

But following the proof of Proposition 6 and Corollary 7, one can easily check rigorously the following result. For $(e_k)_{k\geq 1}$ an orthonormal basis of $L^2(D)$, set $B_t^k = \int_0^t \int_D e_k(x)W(ds,dx)$. For $u_0 \in L^\infty(D)$ and $n \geq 1$, consider the solution (see Pardoux [10]) to

$$u^n(t,x) = u_0(x) + \int_0^t [\partial_{xx}u^n(s,x) + b(u^n(s,x))]ds + \sum_{k=1}^n \int_0^t \sigma(u^n(s,x))e_k(x)dB_s^k.$$

Then if $\sigma(0) = b(0) = 0$, for any $\gamma \in (0,1)$, any $T > 0$,

$$(18) \qquad \mathbb{E}\left[\sup_{[0,T]} ||u^n(t)||_{L^1(D)}^\gamma + \left\{\int_0^T \sum_{k=1}^n \left(\int_{\mathbb{R}^d} \sigma(u^n(t,x))e_k(x)dx\right)^2 ds\right\}^\gamma\right] \leq C_{b,\gamma,T}||u_0||_{L^1(D)}^\gamma,$$

where the constant $C_{b,\gamma,T}$ depends only on γ, T, b (the important fact is that it does not depend on n). Passing to the limit formally in (18) would yield (17). Unfortunately, (18) is not sufficient to ensure that the sequence u^n is compact and tends, up to extraction of a subsequence, to a weak solution u to (16). But this suggests that, when $\sigma(0) = b(0) = 0$, weak solutions to (16) do exist and satisfy (17).

6. Appendix

First, we recall the following results on continuous local martingales.

Lemma 9. *Let $(M_t)_{t\geq 0}$ be a nonnegative continuous local martingale starting from $m \in (0,\infty)$. For all $\gamma \in (0,1)$, there exists a constant C_γ (depending only on γ) such that*

$$\mathbb{E}\left[\sup_{[0,\infty)} M_t^\gamma + \langle M\rangle_\infty^{\gamma/2}\right] \leq C_\gamma m^\gamma.$$

Proof. Classically (see e.g. Revuz-Yor [11, Theorems 1.6 and 1.7 p 181-182]), enlarging the probability space if necessary, there is a standard Brownian motion β such that $M_t = m + \beta_{\langle M\rangle_t}$. Denote now by $\tau_a = \inf\{t \geq 0; \beta_t = a\}$. Since M is nonnegative, we deduce that

$$\langle M\rangle_\infty \leq \tau_{-m} \quad \text{and} \quad \sup_{[0,\infty)} M_t \leq m + \sup_{[0,\tau_{-m})} \beta_s.$$

Thus we just have to prove that $\mathbb{E}[\tau_{-m}^{\gamma/2}] + \mathbb{E}[S_m^\gamma] \leq C_\gamma m^\gamma$, where $S_m = \sup_{[0,\tau_{-m})} \beta_s$.

First, for $x \geq 0$, $P[S_m \geq x] = P[\tau_x \leq \tau_{-m}] = m/(m+x)$. As a consequence, since $\gamma \in (0,1)$,

$$\mathbb{E}[S_m^\gamma] = \int_0^\infty P[S_m^\gamma \geq x]dx = \int_0^\infty \frac{m}{m+x^{1/\gamma}}dx = m^\gamma \int_0^\infty \frac{1}{1+y^{1/\gamma}}dy = C_\gamma m^\gamma.$$

Next, for $t \geq 0$, $P[\tau_{-m} \geq t] = P[\inf_{[0,t]} \beta_s > -m]$. Recalling that $\inf_{[0,t]} \beta_s$ has the same law as $-\sqrt{t}|\beta_1|$, we get $P[\tau_{-m} \geq t] = P[|\beta_1| < m/\sqrt{t}]$. Hence

$$\mathbb{E}[\tau_{-m}^{\gamma/2}] = \int_0^\infty P[\tau_{-m}^{\gamma/2} \geq t]dt = \int_0^\infty P[|\beta_1| < m/t^{1/\gamma}]dt = \int_0^\infty P[(m/|\beta_1|)^\gamma > t]dt = m^\gamma \mathbb{E}\left[|\beta_1|^{-\gamma}\right].$$

This concludes the proof, since $\mathbb{E}\left[|\beta_1|^{-\gamma}\right] < \infty$ for $\gamma \in (0,1)$. $\qquad\square$

Next, we state a technical result on a.s. converging integrals.

Lemma 10. *Let $(K_t)_{t\geq 0}$ be a nonnegative process. Assume that $A_\infty = \int_0^\infty K_t dt < \infty$. Then one may find a sequence $(t_n)_{n\geq 1}$ increasing to infinity such that K_{t_n} tends to 0 in probability as $n \to \infty$.*

Proof. Consider a strictly increasing continuous concave function $\phi : \mathbb{R}_+ \mapsto [0,1]$ such that $\phi(0) = 0$. Using the Jensen inequality, we deduce that

$$\frac{1}{T}\int_0^T \mathbb{E}[\phi(K_s)]ds = \mathbb{E}\left[\frac{1}{T}\int_0^T \phi(K_s)ds\right] \leq \mathbb{E}\left[\phi\left(\frac{1}{T}\int_0^T K_s ds\right)\right] \leq \mathbb{E}\left[\phi\left(\frac{A_\infty}{T}\right)\right],$$

which tends to 0 as $T \to \infty$ by the dominated convergence Theorem. As a consequence, we may find a sequence $(t_n)_{n\geq 1}$ such that $\lim_n \mathbb{E}[\phi(K_{t_n})] = 0$. The conclusion follows. $\qquad\square$

Finally, we prove a technical result on coupling.

Lemma 11. *Consider two probability measures μ, ν on a Polish space $\mathcal{X}$. Let $\Psi : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}_+$ be continuous and assume that $\Psi(x,y) > 0$ for all $x \neq y$. If there is a sequence of $\mathcal{X} \times \mathcal{X}$-valued random variables $(X_n, Y_n)_{n \geq 1}$ such that for all $n \geq 1$, $\mathcal{L}(X_n) = \mu$ and $\mathcal{L}(Y_n) = \nu$ and if $\lim_n \mathbb{E}[\Psi(X_n, Y_n)] = 0$, then $\mu = \nu$.*

Proof. The sequence of probability measures $(\mathcal{L}(X_n, Y_n))_{n \geq 1}$ is obviously tight, so up to extraction of a subsequence, we may assume that (X_n, Y_n) converges in law, to some (X, Y). Of course, $\mathcal{L}(X) = \mu$ and $\mathcal{L}(Y) = \nu$. Since $\Psi \wedge 1$ is continuous and bounded, we deduce that $\mathbb{E}[\Psi(X,Y) \wedge 1] = \lim_n \mathbb{E}[\Psi(X_n, Y_n) \wedge 1] = 0$, whence $\Psi(X, Y) = 0$ a.s. By assumption, this implies that $X = Y$ a.s., so that $\mu = \nu$. $\square$

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